Logics with probabilistic team semantics and the Boolean negation

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Abstract. We study the expressivity and the complexity of various logics in probabilistic team semantics with the Boolean negation. In particular, we study the extension of probabilistic independence logic with the Boolean negation, and a recently introduced logic FOPT. We give a comprehensive picture of the relative expressivity of these logics together with the most studied logics in probabilistic team semantics setting, as well as relating their expressivity to a numerical variant of second-order logic. In addition, we introduce novel entropy atoms and show that the extension of first-order logic by entropy atoms subsumes probabilistic independence logic. Finally, we obtain some results on the complexity of model checking, validity, and satisfiability of our logics.

Keywords: Probabilistic Team Semantics · Model Checking · Satisfiability · Validity · Computational Complexity · Expressivity of Logics

1 Introduction

Probabilistic team semantics is a novel framework for the logical analysis of probabilistic and quantitative dependencies. Team semantics, as a semantic framework for logics involving qualitative dependencies and independencies, was introduced by Hodges [17] and popularised by Väänänen [25] via his dependence logic. Team semantics defines truth in reference to collections of assignments, called *teams*, and is particularly suitable for the formal analysis of properties, such as the functional dependence between variables, that arise only in the presence of multiple assignments. The idea of generalising team semantics to the probabilistic setting can be traced back to the works of Galliani [6] and Hyttinen et al. [18], however the beginning of a more systematic study of the topic dates back to works of Durand et al. [4].

In *probabilistic team semantics* the basic semantic units are probability distributions (i.e., *probabilistic teams*). This shift from set-based to distribution-based

Logic	MC for sentences	SAT	VAL
$\mathrm{FOPT}(\leq^{\delta}_{c})$	PSPACE (Cor. 20)	RE [11, Thm. 5.2]	coRE [11, Thm. 5.2]
$\mathrm{FO}(\perp\!\!\!\perp_{c})$	$\in$ EXPSPACE and NEXPTIME-hard (Thm. 24)	RE (Thm. 26)	coRE (Thm. 26)
$\mathrm{FO}(\sim)$	AEXPTIME[poly] [22, Prop. 5.16, Lem. 5.21]	RE [22, Thm. 5.6]	coRE [22, Thm. 5.6]
$\mathrm{FO}(\approx)$	$\in$ EXPTIME, PSPACE-hard (Thm. 22)	RE (Thm. 26)	coRE (Thm. 26)
$\mathrm{FO}(\sim, \perp\!\!\!\perp_{c})$	$\in$ 3-EXPSPACE, AEXPTIME[poly]-hard (Thm. 25)	RE (Thm. 26)	coRE (Thm. 26)

Table 1. Overview of our results. Unless otherwise noted, the results are completeness results. Satisfiability and Validity are considered for finite structures.

semantics allows probabilistic notions of dependency, such as conditional probabilistic independence, to be embedded in the framework[5]. The expressivity and complexity of non-probabilistic team-based logics can be related to fragments of (existential) second-order logic and have been studied extensively (see, e.g., [7,5,9]). Team-based logics, by definition, are usually not closed under Boolean negation, so adding it can greatly increase the complexity and expressivity of these logics [19,15]. Some expressivity and complexity results have also been obtained for logics in probabilistic team semantics (see below). However, richer semantic and computational frameworks are sometimes needed to characterise these logics.

Metafinite Model Theory, introduced by Grädel and Gurevich [8], generalises the approach of *Finite Model Theory* by shifting to two-sorted structures, which extend finite structures by another (often infinite) numerical domain and weight functions bridging the two sorts. A particularly important subclass of metafinite structures are the so-called $\mathbb{R}$-*structures*, which extend finite structures with the real arithmetic on the second sort. *Blum-Shub-Smale machines* (BSS machines for short) [1] are essentially register machines with registers that can store arbitrary real numbers and compute rational functions over reals in a single time step. Interestingly, Boolean languages which are decidable by a non-deterministic polynomial-time BSS machine coincide with those languages which are PTIME-reducible to the true existential sentences of real arithmetic (i.e., the complexity class $\exists\mathbb{R}$) [2,24].

Recent works have established fascinating connections between second-order logics over $\mathbb{R}$-structures, complexity classes using the BSS-model of computation, and logics using probabilistic team semantics. In [13], Hannula et al. establish that the expressivity and complexity of probabilistic independence logic coincide with a particular fragment of existential second-order logic over $\mathbb{R}$-structures and NP on BSS-machines. In [16], Hannula and Virtema focus on probabilistic inclusion logic, which is shown to be tractable (when restricted to Boolean inputs), and relate it to linear programming.

[5] In [21] Li recently introduced *first-order theory of random variables with probabilistic independence (FOTPI)* whose variables are interpreted by discrete distributions over the unit interval. The paper shows that true arithmetic is interpretable in FOTPI whereas probabilistic independence logic is by our results far less complex.

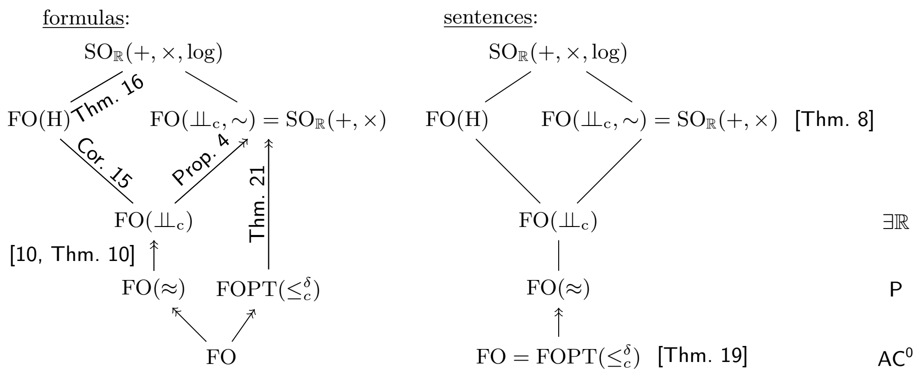

Fig. 1. Landscape of relevant logics as well as relation to some complexity classes. Note that for the complexity classes, finite ordered structures are required. Double arrows indicate strict inclusions.

In this paper, we focus on the expressivity and model checking complexity of probabilistic team-based logics that have access to Boolean negation. We also study the connections between probabilistic independence logic and a logic called $\mathrm{FOPT}(\leq_c^\delta)$, which is defined via a computationally simpler probabilistic semantics [11]. The logic $\mathrm{FOPT}(\leq_c^\delta)$ is the probabilistic variant of a certain team-based logic that can define exactly those dependencies that are first-order definable [20]. We also introduce novel entropy atoms and relate the extension of first-order logic with these atoms to probabilistic independence logic.

See Figure 1 for our expressivity results and Table 1 for our complexity results.

2 Preliminaries

We assume the reader is familiar with the basics in complexity theory [23]. In this work, we will encounter complexity classes PSPACE, EXPTIME, NEXPTIME, EXPSPACE and the class AEXPTIME[poly] together with the notion of completeness under the usual polynomial time many to one reductions. A bit more formally for the latter complexity class which is more uncommon than the others, AEXPTIME[poly] consists of all languages that can be decided by alternating Turing machines within an exponential runtime of $O(2^{n^{O(1)}})$ and polynomially many alternations between universal and existential states. There exist problems in propositional team logic with generalized dependence atoms that are complete for this class [14]. It is also known that truth evaluation of alternating dependency quantified boolean formulae (ADQBF) is complete for this class [14].

2.1 Probabilistic team semantics

We denote first-order variables by x, y, z and tuples of first-order variables by $\mathbf{x}, \mathbf{y}, \mathbf{z}$. For the length of the tuple $\mathbf{x}$, we write $|\mathbf{x}|$. The set of variables that appear in the tuple $\mathbf{x}$ is denoted by $\mathrm{Var}(\mathbf{x})$. A vocabulary τ is a finite set of relation, function, and constant symbols, denoted by R, f, and c, respectively. Each relation symbol R and function symbol f has a prescribed arity, denoted by $\mathrm{Ar}(R)$ and $\mathrm{Ar}(f)$.

Let τ be a finite relational vocabulary such that $\{=\} \subseteq \tau$. For a finite τ-structure $\mathcal{A}$ and a finite set of variables D, an *assignment* of $\mathcal{A}$ for D is a function $s\colon D \to A$. A *team* X of $\mathcal{A}$ over D is a finite set of assignments $s\colon D \to A$.

A *probabilistic team* $\mathbb{X}$ is a function $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers. The value $\mathbb{X}(s)$ is called the *weight* of assignment s. Since zero-weights are allowed, we may, when useful, assume that X is maximal, i.e., it contains all assignments $s\colon D \to A$. The *support* of $\mathbb{X}$ is defined as $\mathrm{supp}(\mathbb{X}) \coloneqq \{s \in X \mid \mathbb{X}(s) \neq 0\}$. A team $\mathbb{X}$ is *nonempty* if $\mathrm{supp}(\mathbb{X}) \neq \varnothing$.

These teams are called probabilistic because we usually consider teams that are probability distributions, i.e., functions $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ for which $\sum_{s\in X} \mathbb{X}(s) = 1$.[6] In this setting, the weight of an assignment can be thought of as the probability that the values of the variables are as in the assignment. If $\mathbb{X}$ is a probability distribution, we also write $\mathbb{X}\colon X \to [0,1]$.

For a set of variables V, the restriction of the assignment s to V is denoted by $s \upharpoonright V$. The *restriction of a team* X to V is $X \upharpoonright V = \{s \upharpoonright V \mid s \in X\}$, and the *restriction of a probabilistic team* $\mathbb{X}$ to V is $\mathbb{X} \upharpoonright V\colon X \upharpoonright V \to \mathbb{R}_{\geq 0}$ where

$$(\mathbb{X} \upharpoonright V)(s) = \sum_{\substack{s' \upharpoonright V = s,\\ s' \in X}} \mathbb{X}(s').$$

If ϕ is a first-order formula, then $\mathbb{X}_\phi$ is the restriction of the team $\mathbb{X}$ to those assignments in X that satisfy the formula ϕ. The weight $|\mathbb{X}_\phi|$ is defined analogously as the sum of the weights of the assignments in X that satisfy ϕ, e.g.,

$$|\mathbb{X}_{\mathbf{x}=\mathbf{a}}| = \sum_{\substack{s \in X,\\ s(\mathbf{x})=\mathbf{a}}} \mathbb{X}(s).$$

For a variable x and $a \in A$, we denote by $s(a/x)$, the modified assignment $s(a/x)\colon D \cup \{x\} \to A$ such that $s(a/x)(y) = a$ if $y = x$, and $s(a/x)(y) = s(y)$ otherwise. For a set $B \subseteq A$, the modified team $X(B/x)$ is defined as the set $X(B/x) \coloneqq \{s(a/x) \mid a \in B, s \in X\}$.

Let $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ be any probabilistic team. Then the probabilistic team $\mathbb{X}(B/x)$ is a function $\mathbb{X}(B/x)\colon X(B/x) \to \mathbb{R}_{\geq 0}$ defined as

$$\mathbb{X}(B/x)(s(a/x)) = \sum_{\substack{t \in X,\\ t(a/x)=s(a/x)}} \mathbb{X}(t) \cdot \frac{1}{|B|}.$$

[6] In some sources, the term probabilistic team only refers to teams that are distributions, and the functions $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ that are not distributions are called *weighted teams*.

If x is a fresh variable, the summation can be dropped and the right-hand side of the equation becomes $\mathbb{X}(s) \cdot \frac{1}{|B|}$. For singletons $B = \{a\}$, we write $X(a/x)$ and $\mathbb{X}(a/x)$ instead of $X(\{a\}/x)$ and $\mathbb{X}(\{a\}/x)$.

Let then $\mathbb{X}\colon X \to [0,1]$ be a distribution. Denote by p_B the set of all probability distributions $d\colon B \to [0,1]$, and let F be a function $F\colon X \to p_B$. Then the probabilistic team $\mathbb{X}(F/x)$ is a function $\mathbb{X}(F/x)\colon X(B/x) \to [0,1]$ defined as

$$\mathbb{X}(F/x)(s(a/x)) = \sum_{\substack{t \in X, \\ t(a/x) = s(a/x)}} \mathbb{X}(t) \cdot F(t)(a)$$

for all $a \in B$ and $s \in X$. If x is a fresh variable, the summation can again be dropped and the right-hand side of the equation becomes $\mathbb{X}(s) \cdot F(s)(a)$.

Let $\mathbb{X}\colon X \to [0,1]$ and $\mathbb{Y}\colon Y \to [0,1]$ be probabilistic teams with common variable and value domains, and let $k \in [0,1]$. The k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, denoted by $\mathbb{X} \sqcup_k \mathbb{Y}$, is the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon Y \to [0,1]$ defined as

$$\mathbb{X} \sqcup_k \mathbb{Y}(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \cap Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \setminus Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \setminus X. \end{cases}$$

3 Probabilistic independence logic with Boolean negation

In this section, we define probabilistic independence logic with Boolean negation, denoted by $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. The logic extends first-order logic with *probabilistic independence atom* $\mathbf{y} \perp\!\!\!\perp_{\mathbf{x}} \mathbf{z}$ which states that the tuples $\mathbf{y}$ and $\mathbf{z}$ are independent given the tuple $\mathbf{x}$. The syntax for the logic $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ over a vocabulary τ is as follows:

$$\phi ::= R(\mathbf{x}) \mid \neg R(\mathbf{x}) \mid \mathbf{y} \perp\!\!\!\perp_{\mathbf{x}} \mathbf{z} \mid {\sim}\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \exists x\phi \mid \forall x\phi,$$

where x is a first-order variable, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are tuples of first-order variables, and $R \in \tau$.

Let ψ be a first-order formula. We denote by $\psi^{\neg}$ the formula which is obtained from $\neg\psi$ by pushing the negation in front of atomic formulas. We also use the shorthand notations $\psi \to \phi := (\psi^{\neg} \vee (\psi \wedge \phi))$ and $\psi \leftrightarrow \phi := \psi \to \phi \wedge \phi \to \psi$.

Let $\mathbb{X}\colon X \to [0,1]$ be a probability distribution. The semantics for the logic is defined as follows:

$\mathcal{A} \models_{\mathbb{X}} R(\mathbf{x})$ iff $\mathcal{A} \models_s R(\mathbf{x})$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} \neg R(\mathbf{x})$ iff $\mathcal{A} \models_s \neg R(\mathbf{x})$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} \mathbf{y} \perp\!\!\!\perp_{\mathbf{x}} \mathbf{z}$ iff $|\mathbb{X}_{\mathbf{xy}=s(\mathbf{xy})}| \cdot |\mathbb{X}_{\mathbf{xz}=s(\mathbf{xz})}| = |\mathbb{X}_{\mathbf{xyz}=s(\mathbf{xyz})}| \cdot |\mathbb{X}_{\mathbf{x}=s(\mathbf{x})}|$ for all $s\colon \mathrm{Var}(\mathbf{xyz}) \to A$.
$\mathcal{A} \models_{\mathbb{X}} {\sim}\phi$ iff $\mathcal{A} \not\models_{\mathbb{X}} \phi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \wedge \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ and $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \vee \psi$ iff $\mathcal{A} \models_{\mathbb{Y}} \phi$ and $\mathcal{A} \models_{\mathbb{Z}} \psi$ for some $\mathbb{Y}, \mathbb{Z}, k$ such that $\mathbb{Y} \sqcup_k \mathbb{Z} = \mathbb{X}$.

$\mathcal{A} \models_{\mathbb{X}} \exists x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(F/x)} \phi$ for some $F\colon X \to p_A$.
$\mathcal{A} \models_{\mathbb{X}} \forall x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(A/x)} \phi$.

The satisfaction relation $\models_s$ above refers to the Tarski semantics of first-order logic. For a sentence ϕ, we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models_{\mathbb{X}_\emptyset} \phi$, where $\mathbb{X}_\emptyset$ is the distribution that maps the empty assignment to 1.

The logic also has the following useful property called *locality*. Denote by $\mathrm{Fr}(\phi)$ the set of the free variables of a formula ϕ.

Proposition 1 (Locality, [4, Prop. 12]). *Let ϕ be any $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)[\tau]$-formula. Then for any set of variables V, any τ-structure $\mathcal{A}$, and any probabilistic team $\mathbb{X}\colon X \to [0,1]$ such that $\mathrm{Fr}(\phi) \subseteq V \subseteq D$,*

$$\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}\upharpoonright V} \phi.$$

In addition to probabilistic conditional independence atoms, we may also consider other atoms. If $\mathbf{x}$ and $\mathbf{y}$ are tuples of variables, then $=(\mathbf{x}, \mathbf{y})$ is a *dependence atom*. If $\mathbf{x}$ and $\mathbf{y}$ are also of the same length, $\mathbf{x} \approx \mathbf{y}$ is a *marginal identity atom*. The semantics for these atoms are defined as follows:

$\mathcal{A} \models_{\mathbb{X}} =(\mathbf{x}, \mathbf{y})$ iff for all $s, s' \in \mathrm{supp}(\mathbb{X})$, $s(\mathbf{x}) = s'(\mathbf{x})$ implies $s(\mathbf{y}) = s'(\mathbf{y})$,
$\mathcal{A} \models_{\mathbb{X}} \mathbf{x} \approx \mathbf{y}$ iff $|\mathbb{X}_{\mathbf{x}=\mathbf{a}}| = |\mathbb{X}_{\mathbf{y}=\mathbf{a}}|$ for all $\mathbf{a} \in A^{|\mathbf{x}|}$.

For two logics L and L' over probabilistic team semantics, we write $L \leq L'$ if for any formula $\phi \in L$, there is a formula $\psi \in L'$ such that $\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}} \psi$ for all $\mathcal{A}$ and $\mathbb{X}$. The equality $\equiv$ and strict inequality $<$ are defined from the above relation in the usual way. The next two propositions follow from the fact that dependence atoms and marginal identity atoms can be expressed with probabilistic independence atoms.

Proposition 2 ([3, Prop. 24]). $\mathrm{FO}(=(\cdot)) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$.

Proposition 3 ([10, Thm. 10]). $\mathrm{FO}(\approx) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$.

On the other hand, omitting the Boolean negation strictly decreases the expressivity as witnessed by the next proposition.

Proposition 4. $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$.

Proof. By Theorems 4.1 and 6.5 of [13], over a fixed universe size, any open formula of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ defines a closed subset of $\mathbb{R}^n$ for a suitable n depending on the size of the universe and the number of free variables. Now, clearly, this cannot be true for all of the formulas of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ as it contains the Boolean negation, e.g., the formula $\sim x \perp\!\!\!\perp_y z$. $\square$

4 Metafinite logics

In this section, we consider logics over $\mathbb{R}$-structures. These structures extend finite relational structures with real numbers $\mathbb{R}$ as a second domain and add functions that map tuples from the finite domain to $\mathbb{R}$.

Definition 5 ($\mathbb{R}$-structures). *Let τ and σ be finite vocabularies such that τ is relational and σ is functional. An* $\mathbb{R}$-structure of vocabulary $\tau \cup \sigma$ *is a tuple $\mathcal{A} = (A, \mathbb{R}, F)$ where the reduct of $\mathcal{A}$ to τ is a finite relational structure, and F is a set that contains functions $f^{\mathcal{A}} \colon A^{\mathrm{Ar}(f)} \to \mathbb{R}$ for each function symbol $f \in \sigma$. Additionally, (i) for any $S \subseteq \mathbb{R}$, if each $f^{\mathcal{A}}$ is a function from $A^{\mathrm{Ar}(f)}$ to S, $\mathcal{A}$ is called an* S-structure, *(ii) if each $f^{\mathcal{A}}$ is a distribution, $\mathcal{A}$ is called a* $d[0,1]$-structure.

Next, we will define certain metafinite logics which are variants of functional second-order logic with numerical terms. The numerical σ-terms i are defined as follows:

$$i ::= f(\mathbf{x}) \mid i \times i \mid i + i \mid \mathrm{SUM}_{\mathbf{y}} i \mid \log i,$$

where $f \in \sigma$ and $\mathbf{x}$ and $\mathbf{y}$ are first-order variables such that $|\mathbf{x}| = \mathrm{Ar}(f)$. The interpretation of a numerical term i in the structure $\mathcal{A}$ under an assignment s is denoted by $[i]^{\mathcal{A}}_s$. We define

$$[\mathrm{SUM}_{\mathbf{y}} i]^{\mathcal{A}}_s := \sum_{\mathbf{a} \in A^{|\mathbf{y}|}} [i]^{\mathcal{A}}_{s(\mathbf{a}/\mathbf{y})}.$$

The interpretations of the rest of the numerical terms are defined in the obvious way.

Suppose that $\{=\} \subseteq \tau$, and let $O \subseteq \{+, \times, \mathrm{SUM}, \log\}$. The syntax for the logic $\mathrm{SO}_{\mathbb{R}}(O)$ is defined as follows:

$$\phi ::= i = j \mid \neg i = j \mid R(\mathbf{x}) \mid \neg R(\mathbf{x}) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \exists x \phi \mid \forall x \phi \mid \exists f \psi \mid \forall f \psi,$$

where i and j are numerical σ-terms constructed using operations from O, $R \in \tau$, x, y, and $\mathbf{x}$ are first-order variables, f is a function variable, and ψ is a $\tau \cup \sigma \cup \{f\}$-formula of $\mathrm{SO}_{\mathbb{R}}(O)$.

The semantics of $\mathrm{SO}_{\mathbb{R}}(O)$ is defined via $\mathbb{R}$-structures and assignments analogous to first-order logic, except for the interpretations of function variables f, which range over functions $A^{\mathrm{Ar}(f)} \to \mathbb{R}$. For any $S \subseteq \mathbb{R}$, we define $\mathrm{SO}_S(O)$ as the variant of $\mathrm{SO}_{\mathbb{R}}(O)$, where the quantification of function variables ranges over $A^{\mathrm{Ar}(f)} \to S$. If the quantification of function variables is restricted to distributions, the resulting logic is denoted by $\mathrm{SO}_{d[0,1]}(O)$. The existential fragment, in which universal quantification over function variables is not allowed, is denoted by $\mathrm{ESO}_{\mathbb{R}}(O)$.

For metafinite logics L and L', we define expressivity comparison relations $L \leq L'$, $L \equiv L'$, and $L < L'$ in the usual way, see e.g. [13]. For the proofs of the following two propositions, see the full version [12] of this paper in ArXiv.

Proposition 6. $\mathrm{SO}_{\mathbb{R}}(\mathrm{SUM}, \times) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$.

Proposition 7. $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$.

5 Equi-expressivity of $\mathbf{FO}(\perp\!\!\!\perp_{\mathbf{c}}, \sim)$ and $\mathbf{SO}_{\mathbb{R}}(+, \times)$

In this section, we show that the expressivity of probabilistic independence logic with the Boolean negation coincides with full second-order logic over $\mathbb{R}$-structures.

Theorem 8. $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$.

We first show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{SO}_{\mathbb{R}}(+, \times)$. Note that by Proposition 7, we have $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$, so it suffices to show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$. We may assume that every independence atom is in the form $\mathbf{y} \perp\!\!\!\perp_{\mathbf{x}} \mathbf{z}$ or $\mathbf{y} \perp\!\!\!\perp_{\mathbf{x}} \mathbf{y}$ where $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are pairwise disjoint tuples. [4, Lemma 25]

Theorem 9. *Let formula $\phi(\mathbf{v}) \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ be such that its free-variables are from $\mathbf{v} = (v_1, \ldots, v_k)$. Then there is a formula $\psi_\phi(f) \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$ with exactly one free function variable such that for all structures $\mathcal{A}$ and all probabilistic teams $\mathbb{X}\colon X \to [0,1]$, $\mathcal{A} \models_{\mathbb{X}} \phi(\mathbf{v})$ if and only if $(\mathcal{A}, f_{\mathbb{X}}) \models \psi_\phi(f)$, where $f_{\mathbb{X}}\colon A^k \to [0,1]$ is a probability distribution such that $f_{\mathbb{X}}(s(\mathbf{v})) = \mathbb{X}(s)$ for all $s \in X$.*

Proof. Define the formula $\psi_\phi(f)$ as follows:

1. If $\phi(\mathbf{v}) = R(v_{i_1}, \ldots, v_{i_l})$, where $1 \leq i_1, \ldots, i_l \leq k$, then $\psi_\phi(f) \coloneqq \forall \mathbf{v}(f(\mathbf{v}) = 0 \vee R(v_{i_1}, \ldots, v_{i_l}))$.
2. If $\phi(\mathbf{v}) = \neg R(v_{i_1}, \ldots, v_{i_l})$, where $1 \leq i_1, \ldots, i_l \leq k$, then $\psi_\phi(f) \coloneqq \forall \mathbf{v}(f(\mathbf{v}) = 0 \vee \neg R(v_{i_1}, \ldots, v_{i_l}))$.
3. If $\phi(\mathbf{v}) = \mathbf{v}_1 \perp\!\!\!\perp_{\mathbf{v}_0} \mathbf{v}_2$, where $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ are disjoint, then
$$\begin{aligned}\psi_\phi(f) \coloneqq \forall \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2(&\mathrm{SUM}_{\mathbf{v}\setminus(\mathbf{v}_0\mathbf{v}_1)} f(\mathbf{v}) \times \mathrm{SUM}_{\mathbf{v}\setminus(\mathbf{v}_0\mathbf{v}_2)} f(\mathbf{v}) = \\ &\mathrm{SUM}_{\mathbf{v}\setminus(\mathbf{v}_0\mathbf{v}_1)} f(\mathbf{v}) \times \mathrm{SUM}_{\mathbf{v}\setminus\mathbf{v}_0} f(\mathbf{v})).\end{aligned}$$
4. If $\phi(\mathbf{v}) = \mathbf{v}_1 \perp\!\!\!\perp_{\mathbf{v}_0} \mathbf{v}_1$, where $\mathbf{v}_0, \mathbf{v}_1$ are disjoint, then
$$\psi_\phi(f) \coloneqq \forall \mathbf{v}_0\mathbf{v}_1(\mathrm{SUM}_{\mathbf{v}\setminus(\mathbf{v}_0\mathbf{v}_1)} f(\mathbf{v}) = 0 \vee \mathrm{SUM}_{\mathbf{v}\setminus(\mathbf{v}_0\mathbf{v}_1)} f(\mathbf{v}) = \mathrm{SUM}_{\mathbf{v}\setminus\mathbf{v}_0} f(\mathbf{v})).$$
5. If $\phi(\mathbf{v}) = \sim \phi_0(\mathbf{v})$, then $\psi_\phi(f) \coloneqq \psi^{\neg}_{\phi_0}(f)$, where $\psi^{\neg}_{\phi_0}$ is obtained from $\neg\psi_{\phi_0}$ by pushing the negation in front of atomic formulas.
6. If $\phi(\mathbf{v}) = \phi_0(\mathbf{v}) \wedge \phi_1(\mathbf{v})$, then $\psi_\phi(f) \coloneqq \psi_{\phi_0}(f) \wedge \psi_{\phi_1}(f)$.
7. If $\phi(\mathbf{v}) = \phi_0(\mathbf{v}) \vee \phi_1(\mathbf{v})$, then
$$\begin{aligned}\psi_\phi(f) \coloneqq\ &\psi_{\phi_0}(f) \vee \psi_{\phi_1}(f) \\ &\vee (\exists g_0 g_1 g_2 g_3 (\forall \mathbf{v} \forall x (x = l \vee x = r \vee (g_0(x) = 0 \wedge g_3(\mathbf{v}, x) = 0)) \\ &\wedge \forall \mathbf{v}(g_3(\mathbf{v}, l) = g_1(\mathbf{v}) \times g_0(l) \wedge g_3(\mathbf{v}, r) = g_2(\mathbf{v}) \times g_0(r)) \\ &\wedge \forall \mathbf{v}(\mathrm{SUM}_x g_3(\mathbf{v}, x) = f(\mathbf{v})) \wedge \psi_{\phi_0}(g_1) \wedge \psi_{\phi_1}(g_2))).\end{aligned}$$
8. If $\phi(\mathbf{v}) = \exists x \phi_0(\mathbf{v}, x)$, then $\psi_\phi(f) \coloneqq \exists g(\forall \mathbf{v}(\mathrm{SUM}_x g(\mathbf{v}, x) = f(\mathbf{v})) \wedge \psi_{\phi_0}(g))$.
9. If $\phi(\mathbf{v}) = \exists x \phi_0(\mathbf{v}, x)$, then
$$\psi_\phi(f) \coloneqq \exists g(\forall \mathbf{v}(\forall x \forall y(g(\mathbf{v}, x) = g(\mathbf{v}, y)) \wedge \mathrm{SUM}_x g(\mathbf{v}, x) = f(\mathbf{v})) \wedge \psi_{\phi_0}(g)).$$

Since the the above is essentially same as the translation in [4, Theorem 14], but extended with the Boolean negation (for which the claim follows directly from the semantical clauses), it is easy to show that $\psi_\phi(f)$ satisfies the claim. $\square$

We now show that $\mathrm{SO}_{\mathbb{R}}(+,\times) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim,)$. By Propositions 3 and 7, $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx) \equiv \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ and $\mathrm{SO}_{\mathbb{R}}(+,\times) \equiv \mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$, so it suffices to show that $\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx)$.

Note that even though we consider $\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$, where only distributions can be quantified, it may still happen that the interpretation of a numerical term does not belong to the unit interval. This may happen if we have a term of the form $\mathrm{SUM}_{\mathbf{x}} i(\mathbf{y})$ where $\mathbf{x}$ contains a variable that does not appear in $\mathbf{y}$. Fortunately, for any formula containing such terms, there is an equivalent formula without them [16, Lemma 19]. Thus, it suffices to consider formulas without such terms.

To prove that $\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx)$, we construct a useful normal form for $\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$-sentences. The following lemma is based on similar lemmas from [4, Lemma, 16] and [16, Lemma, 20]. The proofs of the next two lemmas are in the full version [12] of this paper.

Lemma 10. *Every formula $\phi \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$ can be written in the form $\phi^* := Q_1 f_1 \ldots Q_n f_n \forall \mathbf{x} \theta$, where $Q \in \{\exists, \forall\}$, θ is quantifier-free and such that all the numerical identity atoms are in the form $f_i(\mathbf{uv}) = f_j(\mathbf{u}) \times f_k(\mathbf{v})$ or $f_i(\mathbf{u}) = \mathrm{SUM}_{\mathbf{v}} f_j(\mathbf{uv})$ for distinct f_i,f_j,f_k such that at most one of them is not quantified.*

Lemma 11. *We use the abbreviations $\forall^* x\phi$ and $\phi \to^* \psi$ for the $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx)$-formulas $\sim\exists x \sim\phi$ and $\sim(\phi \wedge \sim\psi)$, respectively. Let $\phi_\exists := \exists \mathbf{y}(\mathbf{x} \perp\!\!\!\perp \mathbf{y} \wedge \psi(\mathbf{x},\mathbf{y}))$ and $\phi_\forall := \forall^* \mathbf{y}(\mathbf{x} \perp\!\!\!\perp \mathbf{y} \to^* \psi(\mathbf{x},\mathbf{y}))$ be $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$-formulas with free variables form $\mathbf{x} = (x_1, \ldots, x_n)$. Then for any structure $\mathcal{A}$ and probabilistic team $\mathbb{X}$ over $\{x_1, \ldots, x_n\}$,*

(i) $\mathcal{A} \models_{\mathbb{X}} \phi_\exists$ iff $\mathcal{A} \models_{\mathbb{X}(d/\mathbf{y})} \psi$ for some distribution $d\colon A^{|\mathbf{y}|} \to [0,1]$,
(ii) $\mathcal{A} \models_{\mathbb{X}} \phi_\forall$ iff $\mathcal{A} \models_{\mathbb{X}(d/\mathbf{y})} \psi$ for all distributions $d\colon A^{|\mathbf{y}|} \to [0,1]$.

Theorem 12. *Let $\phi(p) \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$ be a formula in the form $\phi^* := Q_1 f_1 \ldots Q_n f_n \forall \mathbf{x} \theta$, where $Q \in \{\exists, \forall\}$, θ is quantifier-free and such that all the numerical identity atoms are in the form $f_i(\mathbf{uv}) = f_j(\mathbf{u}) \times f_k(\mathbf{v})$ or $f_i(\mathbf{u}) = \mathrm{SUM}_{\mathbf{v}} f_j(\mathbf{uv})$ for distinct f_i,f_j,f_k from $\{f_1, \ldots, f_n, p\}$. Then there is a formula $\Phi \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx)$ such that for all structures $\mathcal{A}$ and probabilistic teams $\mathbb{X} := p^{\mathcal{A}}$,*

$$\mathcal{A} \models_{\mathbb{X}} \Phi \text{ if and only if } (\mathcal{A}, p) \models \phi.$$

Proof. Define

$$\Phi := \forall \mathbf{x} Q_1^* \mathbf{y}_1 (\mathbf{x} \perp\!\!\!\perp \mathbf{y}_1 \circ_1 Q_2^* \mathbf{y}_2 (\mathbf{x}\mathbf{y}_1 \perp\!\!\!\perp \mathbf{y}_2 \circ_2 Q_3^* \mathbf{y}_3 (\mathbf{x}\mathbf{y}_1\mathbf{y}_2 \perp\!\!\!\perp \mathbf{y}_3 \circ_3 \ldots$$
$$Q_n^* \mathbf{y}_n (\mathbf{x}\mathbf{y}_1 \ldots \mathbf{y}_{n-1} \perp\!\!\!\perp \mathbf{y}_n \circ_n \Theta) \ldots))),$$

where $Q_i^* = \exists$ and $\circ_i = \wedge$, whenever $Q_i = \exists$ and $Q_i^* = \forall^*$ and $\circ_i = \to^*$, whenever $Q_i = \forall$. By Lemma 11, it suffices to show that for all distributions $f_1, \ldots, f_n$, subsets $M \subseteq A^{|\mathbf{x}|}$, and probabilistic teams $\mathbb{Y} := \mathbb{X}(M/\mathbf{x})(f_1/\mathbf{y}_1)\dots(f_n/\mathbf{y}_n)$, we have

$$\mathcal{A} \models_{\mathbb{Y}} \Theta \iff (\mathcal{A}, p, f_1, \dots, f_n) \models \theta(\mathbf{a}) \text{ for all } \mathbf{a} \in M.$$

The claim is shown by induction on the structure of the formula Θ. For the details, see the full ArXiv version [12] of the paper.

1. If θ is an atom or a negated atom (of the first sort), then we let $\Theta := \theta$.
2. Let $\theta = f_i(\mathbf{x}_i) = f_j(\mathbf{x}_j) \times f_k(\mathbf{x}_k)$. Then define
$$\Theta := \exists\alpha\beta((\alpha = 0 \leftrightarrow \mathbf{x}_i = \mathbf{y}_i) \wedge (\beta = 0 \leftrightarrow \mathbf{x}_j\mathbf{x}_k = \mathbf{y}_j\mathbf{y}_k) \wedge \mathbf{x}\alpha \approx \mathbf{x}\beta).$$
The negated case $\neg f_i(\mathbf{x}_i) = f_j(\mathbf{x}_j) \times f_k(\mathbf{x}_k)$ is analogous; just add $\sim$ in front of the existential quantification.
3. Let $\theta = f_i(\mathbf{x}_i) = \mathrm{SUM}_{\mathbf{x}_k} f_j(\mathbf{x}_k\mathbf{x}_j)$. Then define
$$\Theta := \exists\alpha\beta((\alpha = 0 \leftrightarrow \mathbf{x}_i = \mathbf{y}_i) \wedge (\beta = 0 \leftrightarrow \mathbf{x}_j = \mathbf{y}_j) \wedge \mathbf{x}\alpha \approx \mathbf{x}\beta).$$
The negated case $\neg f_i(\mathbf{x}_i) = \mathrm{SUM}_{\mathbf{x}_k} f_j(\mathbf{x}_k\mathbf{x}_j)$ is again analogous.
4. If $\theta = \theta_0 \wedge \theta_1$, then $\Theta = \Theta_0 \wedge \Theta_1$.
5. If $\theta = \theta_0 \vee \theta_1$, then $\Theta := \exists z(z \perp\!\!\!\perp_{\mathbf{x}} z \wedge ((\Theta_0 \wedge z = 0) \vee (\Theta_1 \wedge \neg z = 0)))$.

□

6 Probabilistic logics and entropy atoms

In this section we consider extending probabilistic team semantics with novel entropy atoms. For a discrete random variable X, with possible outcomes $x_1, ..., x_n$ occuring with probabilities $\mathrm{P}(x_1), ..., \mathrm{P}(x_n)$, the Shannon entropy of X is given as:

$$\mathrm{H}(X) := -\sum_{i=1}^{n} \mathrm{P}(x_i) \log \mathrm{P}(x_i),$$

The base of the logarithm does not play a role in this definition (usually it is assumed to be 2). For a set of discrete random variables, the entropy is defined in terms of the vector-valued random variable it defines. Given three sets of discrete random variables X, Y, Z, it is known that X is conditionally independent of Y given Z (written $X \perp\!\!\!\perp Y \mid Z$) if and only if the conditional mutual information $\mathrm{I}(X;Y|Z)$ vanishes. Similarly, functional dependence of Y from X holds if and only if the conditional entropy $H(Y|X)$ of Y given X vanishes. Writing UV for the union of two sets U and V, we note that $\mathrm{I}(X;Y|Z)$ and $H(Y|X)$ can respectively be expressed as $H(ZX) + H(ZY) - H(Z) - H(ZXY)$ and $H(XY) - H(X)$. Thus many familiar dependency concepts over random variables translate into linear equations over Shannon entropies. In what follows, we shortly consider similar information-theoretic approach to dependence and independence in probabilistic team semantics.

Let $\mathbb{X}\colon X \to [0,1]$ be a probabilistic team over a finite structure $\mathcal{A}$ with universe A. Let $\mathbf{x}$ be a k-ary sequence of variables from the domain of $\mathbb{X}$. Let

$P_{\mathbf{x}}$ be the vector-valued random variable, where $P_{\mathbf{x}}(\mathbf{a})$ is the probability that $\mathbf{x}$ takes value $\mathbf{a}$ in the probabilistic team $\mathbb{X}$. The *Shannon entropy* of $\mathbf{x}$ in $\mathbb{X}$ is defined as follows:

$$\mathrm{H}_{\mathbb{X}}(\mathbf{x}) := -\sum_{\mathbf{a}\in A^k} \mathrm{P}_{\mathbf{x}}(\mathbf{a}) \log \mathrm{P}_{\mathbf{x}}(\mathbf{a}). \tag{1}$$

Using this definition we now define the concept of an entropy atom.

Definition 13 (Entropy atom). *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two sequences of variables from the domain of* $\mathbb{X}$*. These sequences may be of different lengths. The* entropy atom *is an expression of the form* $\mathrm{H}(\mathbf{x}) = \mathrm{H}(\mathbf{y})$*, and it is given the following semantics:*

$$\mathcal{A} \models_{\mathbb{X}} \mathrm{H}(\mathbf{x}) = \mathrm{H}(\mathbf{y}) \iff \mathrm{H}_{\mathbb{X}}(\mathbf{x}) = \mathrm{H}_{\mathbb{X}}(\mathbf{y}).$$

We then define *entropy logic* FO(H) as the logic obtained by extending first-order logic with entropy atoms. The entropy atom is relatively powerful compared to our earlier atoms, since, as we will see next, it encapsulates many familiar dependency notions such as dependence and conditional independence. The proof of the theorem is in the full version [12] of this paper.

Theorem 14. *The following equivalences hold over probabilistic teams of finite structures with two distinct constants* 0 *and* 1*:*

1. $=\!(\mathbf{x},\mathbf{y}) \equiv \mathrm{H}(\mathbf{x}) = \mathrm{H}(\mathbf{xy})$.
2. $\mathbf{x} \perp\!\!\!\perp \mathbf{y} \equiv \phi$*, where* ϕ *is defined as*

$$\begin{aligned}\forall z \exists \mathbf{uv}\Big(&\big[z = 0 \to \big(=\!(\mathbf{u},\mathbf{x}) \wedge =\!(\mathbf{x},\mathbf{u}) \wedge =\!(\mathbf{v},\mathbf{xy}) \wedge =\!(\mathbf{xy},\mathbf{v})\big)\big] \wedge \\ &\big[z = 1 \to \big(=\!(\mathbf{u},\mathbf{y}) \wedge =\!(\mathbf{y},\mathbf{u}) \wedge \mathbf{v} = \mathbf{0}\big)\big] \wedge \\ &\big[(z = 0 \vee z = 1) \to \mathrm{H}(\mathbf{u}z) = \mathrm{H}(\mathbf{v}z)\big]\Big),\end{aligned}$$

where $|\mathbf{u}| = \max\{|\mathbf{x}|, \mathbf{y}|\}$ *and* $|\mathbf{v}| = |\mathbf{xy}|$.

Since conditional independence can be expressed with marginal independence, i.e., $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \equiv \mathrm{FO}(\perp\!\!\!\perp)$ [10, Theorem 11], we obtain the following corollary:

Corollary 15. $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \leq \mathrm{FO}(\mathrm{H})$.

It is easy to see at this point that entropy logic and its extension with negation are subsumed by second-order logic over the reals with exponentiation.

Theorem 16. $\mathrm{FO}(\mathrm{H}) \leq \mathrm{ESO}_{\mathbb{R}}(+,\times,\log)$ *and* $\mathrm{FO}(\mathrm{H},\sim) \leq \mathrm{SO}_{\mathbb{R}}(+,\times,\log)$.

Proof. The translation is similar to the one in Theorem 9, so it suffices to notice that the entropy atom $\mathrm{H}(\mathbf{x}) = \mathrm{H}(\mathbf{y})$ can be expressed as

$$\mathrm{SUM}_{\mathbf{z}} f(\mathbf{x},\mathbf{z}) \log f(\mathbf{x},\mathbf{z}) = \mathrm{SUM}_{\mathbf{z}'} f(\mathbf{y},\mathbf{z}') \log f(\mathbf{y},\mathbf{z}').$$

Since SUM can be expressed in $\mathrm{ESO}_{\mathbb{R}}(+,\times,\log)$ and $\mathrm{SO}_{\mathbb{R}}(+,\times,\log)$, we are done. □

7 Logic for first-order probabilistic dependecies

Here, we define the logic $\mathrm{FOPT}(\leq_c^\delta)$, which was introduced in [11].[7] Let δ be a quantifier- and disjunction-free first-order formula, i.e., $\delta ::= \lambda \mid \neg\delta \mid (\delta \wedge \delta)$ for a first-order atomic formula λ of the vocabulary τ. Let x be a first-order variable. The syntax for the logic $\mathrm{FOPT}(\leq_c^\delta)$ over a vocabulary τ is defined as follows:

$$\phi ::= \delta \mid (\delta|\delta) \leq (\delta|\delta) \mid \dot{\sim}\, \phi \mid (\phi \wedge \phi) \mid (\phi \mathbin{\mathbb{V}} \phi) \mid \exists^1 x\phi \mid \forall^1 x\phi.$$

Let $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ be any probabilistic team, not necessarily a probability distribution. The semantics for the logic is defined as follows:

$\mathcal{A} \models_{\mathbb{X}} \delta$ iff $\mathcal{A} \models_s \delta$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} (\delta_0|\delta_1) \leq (\delta_2|\delta_3)$ iff $|\mathbb{X}_{\delta_0\wedge\delta_1}| \cdot |\mathbb{X}_{\delta_3}| \leq |\mathbb{X}_{\delta_2\wedge\delta_3}| \cdot |\mathbb{X}_{\delta_1}|$.
$\mathcal{A} \models_{\mathbb{X}} \dot{\sim}\, \phi$ iff $\mathcal{A} \not\models_{\mathbb{X}} \phi$ or $\mathbb{X}$ is empty.
$\mathcal{A} \models_{\mathbb{X}} \phi \wedge \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ and $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \mathbin{\mathbb{V}} \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ or $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \exists^1 x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(a/x)} \phi$ for some $a \in A$.
$\mathcal{A} \models_{\mathbb{X}} \forall^1 x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(a/x)} \phi$ for all $a \in A$.

Next, we present some useful properties of $\mathrm{FOPT}(\leq_c^\delta)$.

Proposition 17 (Locality, [11, Prop. 3.2]). *Let ϕ be any $\mathrm{FOPT}(\leq_c^\delta)[\tau]$-formula. Then for any set of variables V, any τ-structure $\mathcal{A}$, and any probabilistic team $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ such that $\mathrm{Fr}(\phi) \subseteq V \subseteq D$,*

$$\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}\upharpoonright V} \phi.$$

Over singleton traces the expressivity of $\mathrm{FOPT}(\leq_c^\delta)$ coincides with that of FO. For $\phi \in \mathrm{FOPT}(\leq_c^\delta)$, let ϕ^* denote the FO-formula obtained by replacing the symbols $\dot{\sim}$, $\mathbb{V}$, $\exists^1$, and $\forall^1$ by $\neg, \vee, \exists$, and $\forall$, respectively, and expressions of the form $(\delta_0 \mid \delta_1) \leq (\delta_2 \mid \delta_3)$ by the formula $\neg\delta_0 \vee \neg\delta_1 \vee \delta_2 \vee \neg\delta_3$.

Proposition 18 (Singleton equivalence). *Let ϕ be a $\mathrm{FOPT}(\leq_c^\delta)[\tau]$-formula, $\mathcal{A}$ a τ structure, and $\mathbb{X}$ a probabilistic team of $\mathcal{A}$ with support $\{s\}$. Then $\mathcal{A} \models_{\mathbb{X}} \phi$ iff $\mathcal{A} \models_s \phi^*$.*

Proof. The proof proceeds by induction on the structure of formulas. The cases for literals and Boolean connectives are trivial. The cases for quantifiers are immediate once one notices that interpreting the quantifiers $\exists^1$ and $\forall^1$ maintain singleton supportness. We show the case for $\leq$. Let $\|\delta\|_{\mathcal{A},s} = 1$ if $\mathcal{A} \models_s \delta$, and $\|\delta\|_{\mathcal{A},s} = 0$ otherwise. Then

$$\begin{aligned}
\mathcal{A} \models_{\mathbb{X}} (\delta_0 \mid \delta_1) \leq (\delta_2 \mid \delta_3) &\iff |\mathbb{X}_{\delta_0\wedge\delta_1}| \cdot |\mathbb{X}_{\delta_3}| \leq |\mathbb{X}_{\delta_2\wedge\delta_3}| \cdot |\mathbb{X}_{\delta_1}| \\
&\iff \|\delta_0 \wedge \delta_1\|_{\mathcal{A},s} \cdot \|\delta_3\|_{\mathcal{A},s} \leq \|\delta_2 \wedge \delta_3\|_{\mathcal{A},s} \cdot \|\delta_1\|_{\mathcal{A},s} \\
&\iff \mathcal{A} \models_s \neg\delta_0 \vee \neg\delta_1 \vee \delta_2 \vee \neg\delta_3.
\end{aligned}$$

[7] In [11], two sublogics of $\mathrm{FOPT}(\leq_c^\delta)$, called $\mathrm{FOPT}(\leq^\delta)$ and $\mathrm{FOPT}(\leq^\delta, \perp\!\!\!\perp_c^\delta)$, were also considered. Note that the results of this section also hold for these sublogics.

The first equivalence follows from the semantics of $\leq$ and the second follows from the induction hypotheses after observing that the support of $\mathbb{X}$ is $\{s\}$. The last equivalence follows via a simple arithmetic observation. □

The following theorem follows directly from Propositions 17 and 18.

Theorem 19. *For sentences we have that* $\mathrm{FOPT}(\leq_c^\delta) \equiv \mathrm{FO}$.

For a logic L, we write $\mathsf{MC}(L)$ for the following variant of the model checking problem: given a *sentence* $\phi \in L$ and a structure $\mathcal{A}$, decide whether $\mathcal{A} \models \phi$. The above result immediately yields the following corollary.

Corollary 20. $\mathsf{MC}(\mathrm{FOPT}(\leq_c^\delta))$ *is* PSPACE*-complete.*

Proof. This follows directly from the linear translation of $\mathrm{FOPT}(\leq_c^\delta)$-sentences into equivalent FO -sentences of Theorem 19 and the well-known fact that the model-checking problem of FO is PSPACE-complete. □

The first claim of the next theorem follows from the equi-expressivity of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ and $\mathrm{SO}_{\mathbb{R}}(+, \times)$, and the fact that every $\mathrm{FOPT}(\leq_c^\delta)$ formula can be translated to $\mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$, a sublogic of $\mathrm{SO}_{\mathbb{R}}(+, \times)$. For the details and the proof of the second claim, see the full version [12] of this paper.

Theorem 21. $\mathrm{FOPT}(\leq_c^\delta) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ *and* $\mathrm{FOPT}(\leq_c^\delta)$ *is non-comparable to* $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ *for open formulas.*

8 Complexity of satisfiability, validity and model checking

We now define satisfiability and validity in the context of probabilistic team semantics. Let $\phi \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim, \approx)$. The formula ϕ *is satisfiable in a structure* $\mathcal{A}$ if $\mathcal{A} \models_{\mathbb{X}} \phi$ for some probabilistic team $\mathbb{X}$, and ϕ *is valid in a structure* $\mathcal{A}$ if $\mathcal{A} \models_{\mathbb{X}} \phi$ for all probabilistic teams $\mathbb{X}$ over $\mathrm{Fr}(\phi)$. The formula ϕ is *satisfiable* if there is a structure $\mathcal{A}$ such that ϕ is satisfiable in $\mathcal{A}$, and ϕ is *valid* if ϕ is valid in $\mathcal{A}$ for all structures $\mathcal{A}$.

For a logic L, the satisfiability problem $\mathsf{SAT}(L)$ and the validity problem $\mathsf{VAL}(L)$ are defined as follows: given a formula $\phi \in L$, decide whether ϕ is satisfiable (or valid, respectively).

Theorem 22. $\mathsf{MC}(\mathrm{FO}(\approx))$ *is in* $\mathsf{EXPTIME}$ *and* PSPACE*-hard.*

Proof. First note that $\mathrm{FO}(\approx)$ is clearly a conservative extension of FO, as it is easy to check that probabilistic semantics and Tarski semantics agree on first-order formulas over singleton traces. The hardness now follows from this and the fact that model checking problem for FO is PSPACE-complete.

For upper bound, notice first that any $\mathrm{FO}(\approx)$-formula ϕ can be reduced to an almost conjunctive formula ψ^* of $\mathrm{ESO}_R(+, \leq, \mathrm{SUM})$ [16, Lem, 17]. Then the desired bounds follow due to the reduction from Proposition 3 in [16]. The mentioned reduction yields families of systems of linear inequalities $\mathcal{S}$ from a structure $\mathcal{A}$ and assignment s such that a system $S \in \mathcal{S}$ has a solution if and only if $\mathcal{A} \models_s \phi$. For a FO($\approx$)-formula ϕ, this transition requires exponential time and this yields membership in EXPTIME.

□

This lemma is used to prove the upper-bounds in the next three theorems. See the full version [12], for the proofs of the lemma and the theorems.

Lemma 23. *Let $\mathcal{A}$ be a finite structure and $\phi \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. Then there is a first-order sentence $\psi_{\phi,\mathcal{A}}$ over vocabulary $\{+, \times, \leq, 0, 1\}$ such that ϕ is satisfiable in $\mathcal{A}$ if and only if $(\mathbb{R}, +, \times, \leq, 0, 1) \models \psi_{\phi,\mathcal{A}}$.*

Theorem 24. MC($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$) *is in* EXPSPACE *and* NEXPTIME*-hard.*

Theorem 25. MC($\mathrm{FO}(\sim, \perp\!\!\!\perp_{\mathrm{c}})$) $\in$ 3-EXPSPACE *and* AEXPTIME[poly]*-hard.*

Theorem 26. SAT($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$) *is* RE-, VAL($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$) *is* coRE*-complete.*

Corollary 27. SAT($\mathrm{FO}(\approx)$) *and* SAT($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$) *are* RE- *and* VAL($\mathrm{FO}(\approx)$) *and* VAL($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$) *are* coRE*-complete.*

Proof. The lower bound follows from the fact that FO($\approx$) and FO($\perp\!\!\!\perp_{\mathrm{c}}$) are both conservative extensions of FO. We obtain the upper bound from the previous theorem, since FO($\perp\!\!\!\perp_{\mathrm{c}}, \sim$) includes both FO($\approx$) and FO($\perp\!\!\!\perp_{\mathrm{c}}$). □

9 Conclusion

We have studied the expressivity and complexity of various logics in probabilistic team semantics with the Boolean negation. Our results give a quite comprehensive picture of the relative expressivity of these logics and their relations to numerical variants of (existential) second-order logic. An interesting question for further study is to determine the exact complexities of the decision problems studied in Section 8. Furthermore, dependence atoms based on various notions of entropy deserve further study, as do the connections of probabilistic team semantics to the field of information theory.

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